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Flavour from partially resolved singularities

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Abstract

In this letter we study topological open string field theory on D–branes in a IIB background given by non compact CY geometries $\mathcal{O}(n) \oplus \mathcal{O}(-2-n)$ on $\mathbf{P}^1$ with a singular point at which an extra fiber sits. We wrap N D5-branes on $\mathbf{P}^1$ and M effective D3-branes at singular points, which are actually D5–branes wrapped on a shrinking cycle. We calculate the holomorphic Chern-Simons partition function for the above models in a deformed complex structure and find that it reduces to multi–matrix models with flavour. These are the matrix models whose resolvents have been shown to satisfy the generalized Konishi anomaly equations with flavour. In the $n = 0$ case, corresponding to a partial resolution of the A_2 singularity, the quantum superpotential in the $\mathcal{N} = 1$ unitary SYM with one adjoint and M fundamentals is obtained. The $n = 1$ case is also studied and shown to give rise to two–matrix models which for a particular set of couplings can be exactly solved. We explicitly show how to solve such a class of models by a quantum equation of motion technique.

1 Introduction

A type II background of the form $\mathbb{R}^{1,3} \times X$, where X is Calabi-Yau threefold, containing BPS branes and fluxes, generically produces low energy effective theories with $N = 1$ supersymmetry. While the relation between open/closed string moduli and effective gauge theories ("geometric engineering") is quite well understood in the particular case of $N = 2$ supersymmetry, the $N = 1$ case still lacks a complete understanding. For this reason, the study of the dynamics of branes in Calabi-Yau manifolds has attracted a lot of attention both for its theoretical and phenomenological applications, e.g. [1]. For instance, considering D–branes wrapped around two–cycles in a non–compact CY one can link the superpotential of the $N = 1$ supersymmetric gauge theories living on the space–filling branes to the deformation of the CY geometry [2, 3].

In previous papers [4, 5], building up on the existing literature (see e.g. [6] and references therein), we analyzed type IIB superstring theory background with space filling D5–branes wrapped around two–cycles of a non–compact Calabi–Yau threefold. We explicitly showed that, upon topological twist, the theory reduces to a (multi–)matrix model whose potential describes deformations of complex structures in the singular Calabi–Yau threefolds. Some geometrical properties of these spaces have been studied in [7]. In this paper we extend this analysis by including flavour. Phrasing it in another way, we geometric engineer $N = 1$ (supersymmetric gauge) theories containing fields in the fundamental representation of the gauge group, by introducing a suitable brane background.

More precisely we study topological open string field theory on branes in a IIB background (for the relation between topological strings and superstrings in this context, see [8]), where X is a non–compact CY given by $\mathcal{O}(n) \oplus \mathcal{O}(-2-n)$ on $\mathbf{P}^1$ with a singular point at which an extra fiber sits. We wrap N space–filling D5-branes on $\mathbf{P}^1$ and complete the configuration with M 'effective' D3-branes stuck at the singular point. In the case $n = 0$ we show that the latter can be actually interpreted as D5–branes wrapped around a two–cycle which is subsequently shrunk to the singular point. This requires a partially resolved geometrical set up, which we describe in detail (see the Appendix). While the D5–brane sector gives rise, as in the smooth case, to the superpotential (and, from a geometrical point of view, describes deformations of the smooth CY complex structure), the effective D3-brane sector gives rise to a novel part of the spectrum, the corresponding superpotential data being related to *linear* deformations of the CY complex structure along the extra fiber.

We calculate the partition function for the above models by direct computation of the holomorphic Chern-Simons partition function in a deformed complex structure and find that it reduces to multi–matrix models with flavour [9, 10, 11]. These are the matrix models whose resolvents have been shown to satisfy the generalized Konishi anomaly equations with flavour [12]. In the $n = 0$ case, the quantum superpotential in the $\mathcal{N} = 1$ $U(N_c)$ gauge theory with one adjoint and N_f fundamentals is obtained. The $n = 1$ case is studied in detail and shown to give rise to matrix models which for a particular set of couplings turn out to be solvable. In general the flavour can be integrated out and one obtains a matrix model with a polynomial plus a logarithmic potential. We give explicit examples of how to solve the latter with the technique of the quantum equations of motion. For recent related works, see [13].

The paper is organized as follows. In section 2 we study the reduction of the holomorphic Chern-Simons on the vector bundle $\mathcal{N} = \mathcal{O}(n) \oplus \mathcal{O}(-2-n)$ on $\mathbf{P}^1$, augmented by an extra

fiber, to a two dimensional theory over $\mathbf{P}^1$. In section 3 we show that the calculation of the relevant partition function reduces to (multi)matrix integrals with flavour, and produce some explicit examples. In section 4 we study and solve a specific class of examples of matrix models arising in the above set-up. Section 5 is left for few concluding remarks. In Appendix we work out in detail the geometry of of a partially resolved A_2 singularity which is relevant for the $n = 0$ case.

2 Holomorphic Chern-Simons and two–dimensional gauge theories

Let us consider the non compact CY geometries given by the total space of the vector bundle $\mathcal{N} = \mathcal{O}(n) \oplus \mathcal{O}(-2-n)$ on $\mathbf{P}^1$ augmented by an extra fiber at a singular point of the $\mathbf{P}^1$ and let us denote this space [1] by CY_n. For $n = 0$ this space is the partial resolution of an A_2 singularity, as it is described in detail in the Appendix. The other cases with $n > 0$ are a generalization thereof.

We consider type IIB theory on $R^{1,3} \times CY_n$ with N D5-branes along $R^{1,3} \times \mathbf{P}^1$ and M 3-branes along $R^{1,3}$, the latter being stuck at a singular point in $\mathbf{P}^1$ where the extra fiber sits. The Calabi–Yau threefold can be topologically written as $\mathcal{N} \vee \mathbf{C}^2$, where $\vee$ is the reduced union, i.e. the disjoint union of two spaces with a base point of each identified. In the case $n = 0$ we interpret this geometric background in the following way. We start with a resolution of an A_2 singularity, see Appendix, and wrap N D5–branes on one cycle and M and the other. When we blow down the latter, the M D5–branes wrapped around the shrunk cycle will appear as effective D3–branes stuck at the singular point in the remaining $\mathbf{P}^1$: they cannot vibrate along the base, while they are free to oscillate along the extra fiber[2]. As we said, for generic n we single out a point on $\mathbf{P}^1$ and add an extra fiber to render it singular, but the interpretation as blow-down of a smooth cycle is not as evident as for $n = 0$.

In this paper we would like to show that, upon topological twist, the superpotential of this theory can be calculated by means of the second quantized topological string theory.

The latter is given by the holomorphic Chern-Simons theory [14] (see also the lectures [15])

$$S(\mathcal{A}_T) = \frac{1}{g_s^2} \int_{CY_n} \Omega \wedge Tr_{N+M} \left(\frac{1}{2} \mathcal{A}_T \wedge \bar{\partial} \mathcal{A}_T + \frac{1}{3} \mathcal{A}_T \wedge \mathcal{A}_T \wedge \mathcal{A}_T \right) \tag{1}$$

where $\mathcal{A}_T \in T^{(0,1)}(CY_n)$ with full Chan-Paton index $N + M$. The total string field $\mathcal{A}_T$ can be expanded as

$$\mathcal{A}_T = \begin{pmatrix} \mathcal{A} & \mathcal{X} \\ \tilde{\mathcal{X}} & 0 \end{pmatrix}$$

[1] More general geometries built as the total space of a rank 2 holomorphic vector bundle over a generic Riemann surface could be considered along the lines of [4], but we will not do it here.

[2] In this paper we are using a simple (and probably too poor) language in which B-branes are complex submanifolds of the target space. In a more sophisticated approach isolated fibers over singular points should be replaced by skyscraper sheaves and, in general, B–branes should be embedded as complexes in derived categories of coherent sheaves, see [17] for a review. We show that our approach nevertheless captures some relevant topological information.

where $\mathcal{A}$ is the string field for the 5-5 sector and $(\mathcal{X}, \tilde{\mathcal{X}})$ for the 5-3 and 3-5 open strings. The 3-3 sector is irrelevant to us (anyway, see next footnote).

The action (1) reduces to

$$S(\mathcal{A}_T) = \frac{1}{g_s^2}\int_{CY_n} \Omega \wedge \left[Tr \left(\frac{1}{2}\mathcal{A} \wedge \bar{\partial}\mathcal{A} + \frac{1}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right)\right.$$

$$\left. + \frac{1}{2}\sum_{I=1}^{M}\left(\bar{D}_{\mathcal{A}}\tilde{\mathcal{X}}_I \wedge \mathcal{X}^I + \tilde{\mathcal{X}}_I \wedge \bar{D}_{\mathcal{A}}\mathcal{X}^I\right)\right] \tag{2}$$

where gauge indices are not explicitly shown, $I = 1, \ldots, M$ is the flavour index and $D_{\mathcal{A}}$ is the covariant derivative in the (anti-)fundamental representation.

The reduction of the open string field to the D5-brane world-volume $\mathbf{P}^1$ is obtained via an auxiliary invertible bilinear form on $\mathcal{N} \otimes \bar{\mathcal{N}}$ which we denote by K and its associated Chern connection $\Gamma_{\bar{z}} = K^{-1}\partial_{\bar{z}}K$. The reduction condition for the 5-5 sector is $(D_\Gamma \mathcal{A})^{\mathcal{N}} = 0$, where D_Γ is the covariant derivative w.r.t. Γ and the $\mathcal{N}$ index denotes projection along the fiber directions. The reduction conditions for the 5-3 and 3-5 sector read $(D_\Gamma\mathcal{X})^{\mathcal{N}} = 0$ and $i_{\partial_{\bar{z}}}\mathcal{X}|_{\mathbf{P}^1} = 0$, as well as $(D_\Gamma\tilde{\mathcal{X}})^{\mathcal{N}} = 0$ and $i_{\partial_{\bar{z}}}\tilde{\mathcal{X}}|_{\mathbf{P}^1} = 0$. The last condition specifies that the D3-branes are stuck at the singular point in $\mathbf{P}^1$ and therefore their oscillations along $T\mathbf{P}^1$ are inhibited [3].

The reduction of the 5-5 sector has been already discussed in [4] and will not be repeated here. The reduction of the flavour sector can be carried out with the same technique. Let $X^I_{\bar{i}} = K_{\bar{i}j}Q^{jI}$ and analogously $\tilde{X}_{\bar{i}I} = K_{\bar{i}j}\tilde{Q}^j_I$, where $\mathcal{X} = X_{\bar{i}}dw^{\bar{i}}$ and $\tilde{\mathcal{X}} = \tilde{X}_{\bar{i}}dw^{\bar{i}}$ solve the above reduction conditions. The reduction of the Lagrangian density to the $\mathbf{P}^1$ for the 5-3 sector follows the same logic as for the 5-5 sector. The proper pullback to the base is performed patch by patch with the help of K as contraction of the hCS (3,3)-form Lagrangian by the two bi-vector fields $k = \frac{1}{2}\epsilon_{ij}K^{i\bar{l}}K^{j\bar{k}}\frac{\partial}{\partial\bar{w}^l}\frac{\partial}{\partial\bar{w}^k}$ and $\rho = \frac{1}{2}\epsilon^{ij}\frac{\partial}{\partial w^i}\frac{\partial}{\partial w^j}$.

The resulting $(1,1)$-form Lagrangian density reads

$$L_{fl.red.} = \frac{1}{2}\epsilon_{ij}\tilde{Q}^i_I D_{\bar{z}}Q^{jI} - \frac{1}{2}\epsilon_{ij}D_{\bar{z}}\tilde{Q}^i_I Q^{jI} \tag{3}$$

which is independent on K. This proves that our reduction mechanism is well defined. The action (3) was given in [18] in a similar context and is always a $\beta\gamma$-system. Combining with the 5-5 sector, the total reduced action then reads

$$L_{red.} = \frac{1}{2}\epsilon_{ij}\mathrm{Tr}\phi^i D_{\bar{z}}\phi^j + \frac{1}{2}\epsilon_{ij}\tilde{Q}^i_I D_{\bar{z}}Q^{jI} - \frac{1}{2}\epsilon_{ij}D_{\bar{z}}\tilde{Q}^i_I Q^{jI} \tag{4}$$

It is straightfoward to generalize the gauge fixing procedure for the 5-5 sector, see [5], to the total system to show that the partition function of the theory above reduces to matrix integrals over the $\partial_{\bar{z}}$ zero–modes of the fields.

[3] Actually, the 3-3 sector would appear as the C component in $\mathcal{A}_T = \begin{pmatrix} \mathcal{A} & \mathcal{X} \\ \tilde{\mathcal{X}} & C \end{pmatrix}$. It would modify the action by $\Delta S = \frac{1}{g_s^2}\int_{CY_n}\Omega \wedge \left[Tr_M\left(\frac{1}{2}C\wedge\bar{\partial}C + \frac{1}{3}C\wedge C\wedge C\right) + \tilde{X}\wedge C\wedge X\right]$. Upon the reduction condition for the 3-3 sector $i_{\partial_{\bar{z}}}C|_{\mathbf{P}^1} = 0$, we see that the last two terms give vanishing contribution (neither C nor the X's have a $d\bar{z}$ component to complete a top-form on CY_n), therefore the 3-3 sector decouples.

3 Deformed complex structures and matrix models

The deformation of the complex structure of the singular spaces we are considering can be split in two operations, namely the deformation of the smooth part and the deformation of the extra fiber at the singular point.

As far as the smooth part is concerned, the deformed complex structures to which we confine are of the form obtained by glueing the north and south patches of the fibers above the sphere as

$$\omega_N^1 = z_S^{-n}\omega_S^1, \quad \text{and} \quad \omega_N^2 = z_S^{2+n}\left[\omega_S^2 + \partial_{\omega^1} B\left(z_S, \omega_S^1\right)\right] \tag{5}$$

This, as is well–known, preserves the Calabi-Yau property of the six manifold. As widely discussed in [4] the glueing conditions (5) gets promoted to the glueing conditions for the 5-5 sector, that is the chiral adjoints.

Now let us deal with the analogous deformation for the 5-3 and 3-5 sectors. Let $\hat{P}$ be the point on $\mathbf{P}^1$ where the extra fiber sits and let (x^1, x^2) be the coordinates along the latter. Before the complex structure deformation, the extra fiber glueing is given by

$$x_N^1 = \hat{z}_S^{-n} x_S^1, \quad \text{and} \quad x_N^2 = \hat{z}_S^{2+n} x_S^2.$$

The complex structure deformations we confine to for this sector are linear and are described by the glueing conditions

$$x_N^1 = \hat{z}_S^{-n} x_S^1, \quad \text{and} \quad x_N^2 = \hat{z}_S^{2+n}\left[x_S^2 + M\left(\hat{z}_S, \omega_S^1\right) x_S^1\right], \tag{6}$$

where M is locally analytic on $\mathbf{C} \times (U_N \cap U_S)$. This can be cast in the form

$$M(z,\omega) = \sum_{d=1}^{\infty} \sum_{k=0}^{nd+2} m_d^k z^{-k-1} \omega^d. \tag{7}$$

Notice however that only a subset of the parameters m_d^k parameterize actual deformations of the complex structure, since only a part of them cannot be reabsorbed by a local reparametrization.

The deformed glueing condition (6) is coupled to the 3-5 sector of the topological open string field since the 3-branes only vibrate transversely along the extra-fiber.

Note that we are obtaining different string backgrounds on our geometry, by considering the smooth variety (5) and 'attaching' to it additional fibres, eq. (6). The function M then generates the variation of the complex structure of the CY along the singular fiber.

This deformed geometry (6) can be implemented in the reduction of the open string field in a way much similar to the one followed for the pure 5-5 sector in the smooth case. This is done by promoting (5-6) to the glueing conditions of the reduced string field components ϕ^i, X^i and $\tilde{X}^i$ with a patch by patch singular field redefinition which reabsorbs the deformation terms (B, M). In these singular coordinates the fields glue linearly and we can apply the results of the previous section, obtaining in this way the Lagrangian (4) in the singular field coordinates. Going back to the regular coordinates, one gets the action

$$S_{red.} = \int_{\mathbf{P}^1} \mathrm{Tr}\left[\phi^2 D_{\bar{z}}\phi^1 + \tilde{Q}_I^2 D_{\bar{z}} Q^{1I} + Q^{I2} D_{\bar{z}} \tilde{Q}_I^1\right]$$

$$+\oint_{aequator} dz \left[\mathrm{Tr} B(\phi^1, z) + \tilde{Q}^1_I M(\phi^1, z) Q^{1I}\right]. \tag{8}$$

The partition function of the latter theory can be calculated as in [5] and one gets as a result a multi matrix–model of vector type, namely with additional interactions with flavours. The fields ϕ^1 and $(Q^1, \tilde{Q}^1)$ contribute only through their $\partial_{\bar{z}}$ zero–modes. Under the above glueing prescription we expand $\phi^1(z) = \sum_{i=0}^n z^i X_i$ and analogously $Q^{1I}(z) = \sum_{i=0}^n z^i q_i^I$ and $\tilde{Q}^1_I(z) = \sum_{i=0}^n z^i \tilde{q}_{iI}$, where X_i, q_i^I and $\tilde{q}_{iI}$ are matrix and vector constant coefficients.

Specifically the partition function, after the above calculations, reads

$$Z_n = \int \prod_{i=0}^n dX_i dq_i d\tilde{q}_i \, \mathrm{e}^{\frac{1}{g_s^2}\left(\mathrm{Tr}\mathcal{W}(X) + \tilde{q}_i^I \mathcal{M}(X)_{ij} q_{Ij}\right)} \tag{9}$$

where

$$\mathcal{W}(X) = \oint dz \, B\left(\sum_{i=0}^n z^i X_i, z\right)$$

is the 5-5 contribution already obtained in [4] and

$$\mathcal{M}(X)_{ij} = \oint dz \, z^{i+j} M\left(\sum_{k=0}^n z^k X_k, z\right)$$

represents the 5-3/3-5 coupling.

We are therefore finding in the general case a multi matrix model with flavour symmetry. As explained in [19], these matrix models are related to the quantum superpotential of a $\mathcal{N} = 1$ SYM with $(n+1)M$ chiral multiplets in the fundamental/anti-fundamental and $n+1$ in the adjoint representation.

Let us specify a couple of examples which turn out to be interesting.

3.1 One chiral in the adjoint ($n = 0$ case)

In particular, if $n = 0$ and the CY manifold is the total space of $[\mathcal{O}(-2)_{\mathbf{P}^1} \vee \mathbf{C}] \times \mathbf{C}$ we have a single set of constant zero–modes. Choosing

$$B(\omega^1, z) = \frac{1}{z}\mathcal{W}(\omega^1) \quad \text{and} \quad M(\omega^1, z) = \frac{1}{z}\mathcal{M}(\omega^1)$$

we find

$$Z_0 = \int dX dq d\tilde{q} \, \mathrm{e}^{\frac{1}{g_s^2}\left(\mathrm{Tr}\mathcal{W}(X) + \tilde{q}^I \mathcal{M}(X) q_I\right)}. \tag{10}$$

This partition function is then an extended matrix model with vectorial entries of the same type as the ones first considered in [9], which gives rise, in the large N expansion, to the quantum superpotential for $\mathcal{N} = 1$ with M flavour. Actually a deeper analysis of these models started soon after [11] culminated in [12], where it was shown that the resolvent of the above enriched matrix model (10) solves the generalized Konishi anomaly equations of the corresponding four dimensional gauge theory. It is natural therefore to conjecture that the same is true for the other matrix models (9) that we have just obtained above.

In particular, if $\mathcal{M}(\omega^1) = \omega^1 - m$ and $\mathcal{W}'(\omega^1) = (\omega^1)^N + \ldots$, the correct SW curve

$$y^2 = [\mathcal{W}'(x)]^2 + \Lambda^{2N-N_f}(x-m)^{N_f}$$

is recovered [10]. The above SW curve should be related to the deformed partially resolved geometry we are considering.

3.2 Two adjoint chirals ($n = 1$ case)

As a further example, let us discuss the result we obtain in the $n = 1$ case. Let us denote by Φ_0 and Φ_1 the two adjoint chiral superfields, then the formulas for the superpotential and for the mass term read

$$\begin{aligned}
\mathcal{W}(\Phi_0, \Phi_1) &= \sum_{d=1}^{\infty}\sum_{k=0}^{d} t_d^k \sum_{\substack{i_1,\ldots i_d=0,1\\ i_1+\cdots+i_d=k}} \Phi_{i_1}\ldots\Phi_{i_d} \\
\mathcal{M}(\Phi_0, \Phi_1) &= \sum_{d=1}^{\infty}\sum_{k=0}^{d} \begin{pmatrix} m_d^k & m_d^{k+1} \\ m_d^{k+1} & m_d^{k+2} \end{pmatrix} \sum_{\substack{i_1,\ldots i_d=0,1\\ i_1+\cdots+i_d=k}} \Phi_{i_1}\ldots\Phi_{i_d}\ .
\end{aligned} \tag{11}$$

Note, for later use, that it is possible to produce a superpotential and a mass term of the form (with $X \equiv \Phi_0$, $Y \equiv \Phi_1$)

$$\begin{aligned}
\mathcal{W}(X, Y) &= V(X) + t'Y^2 + cXY \\
\mathcal{M}(X, Y) &= \begin{pmatrix} \mathcal{M}_1(X) & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned} \tag{12}$$

by considering the following geometric deformation terms

$$\begin{aligned}
B(z, \omega) &= \frac{1}{z}V(\omega) + \frac{t'}{z^3}\omega^2 + \frac{c}{2z^2}\omega^2 \\
M(z, \omega) &= \frac{1}{z}\mathcal{M}_1(\omega)
\end{aligned} \tag{13}$$

4 Solving the matrix model

Matrix models of the type (10) can be exactly solved. They are equivalent to one–matrix models with a polynomial + logarithmic potential. In the following we show that the technology of two–matrix models can be profitably used to solve them. This goes as follows. Let us consider, for simplicity, the case in which in (10) there is only one flavour (the extension to many flavours is straightforward). The integral over $\tilde{q}$ and q can be explicitly carried out and produces the determinant of $\mathcal{M}$ to the power -1. This can be written as the exponent of $-\mathrm{Tr}\ln\mathcal{M}$ (for simplicity we set $g_s = 1$, which is equivalent to rescaling the linear couplings in $\mathcal{W}$ and $\mathcal{M}$). Therefore (10) is equivalent to

$$Z_0 = \int dX e^{\mathrm{Tr}(\mathcal{W}(X) - \ln\mathcal{M}(X))}. \tag{14}$$

where

$$\mathcal{W}(X)=\sum_{k=1}^{p} t_k X^k, \qquad \mathcal{M}(X)=\sum_{k=1}^{q} s_k X^k. \tag{15}$$

In order to be able to exploit the powerful technology of two–matrix models we couple this model to a Gaussian one[4] with a bilinear coupling (see section 3.2 for a derivation of this model):

$$Z_0'=\int dXdY \mathrm{e}^{\mathrm{Tr}\left(\mathcal{W}(X)-\ln\mathcal{M}(X)+cXY+t'Y^2\right)} \tag{16}$$

After performing the path integral with the orthogonal polynomials method we will decouple the Gaussian model by setting $c=0$ (see, for instance, [4]). By means of the orthogonal polynomials we can perform the path integral explicitly and reduce the problem to that of solving the quantum equations of motion. Once this is done, the model is completely solved because there exists a precise algorithm (based on the flow equations of the Toda lattice hierarchy) to calculate all the correlators.

Therefore the basic step in order to solve the model (16) is to solve the quantum equations of motion. In this case they take the form

$$P^{\circ}(1)+\mathcal{W}'(Q(1))+cQ(2)=\frac{\mathcal{M}'(Q(1))}{\mathcal{M}(Q(1))}, \tag{17}$$

$$cQ(1)+2t'Q(2)+\widetilde{\mathcal{P}}^{\circ}(2)=0, \tag{18}$$

where a prime denotes functional derivative with respect to the (matrix) entry. $Q(1)$, $Q(2)$, $P^{\circ}(1)$, $\widetilde{\mathcal{P}}^{\circ}(2)$ represent, in the basis of the orthogonal polynomials, multiplication by the eigenvalues λ_1,λ_2 of X and Y, respectively, and the derivatives with respect to the same parameters. Eqs. (17, 18) can be considered as the quantum analog of the classical equations of motion. The difference with the classical equations of motion of the original matrix model is that, instead of the $N\times N$ matrices M_1 and M_2, here we have infinite $Q(1)$ and $Q(2)$ matrices together with the quantum deformation terms represented by $P^{\circ}(1)$ and $\widetilde{\mathcal{P}}^{\circ}(2)$, respectively. From the quantum equations of motion it follows that $Q(1)$ and $Q(2)$ are Jacobi matrices, that is they have a diagonal band structure and can be parameterized as follows.

$$Q(1)=\sum_{i=0}^{\infty}(E_{i,i+1}+a_0(i)E_{i,i}+a_1(i+1)E_{i+1,i}) \tag{19}$$

$$Q(2)=\sum_{i}\left(R(i+1)E_{i,i+1}+\sum_{l=0}^{\infty}\frac{b_l(i)}{R(i+1)\dots R(i+l)}E_{i,i+l}\right) \tag{20}$$

where $(E_{i,j})_{k,l}=\delta_{i,k}\delta_{j,l}$, $R(i+1)\equiv h_{i+1}/h_i$.

To show something explicit without cluttering the paper with formulas, we choose a simple example

$$\mathcal{W}(X)=t_2X^2+t_3X^3, \qquad \mathcal{M}(X)=s_0+s_2X^2 \tag{21}$$

[4]Instead of a Gaussian model we could take any polynomial one, but the analysis would be much more complicated.

and write down explicit formulas only for the genus 0 case: this corresponds to the lattice fields being independent of the lattice increment and to the replacement $n \to x = n/N$. Moreover we rescale the couplings as follows: $t_2 \to -N/(2g_s), t_3 \to -Ng/g_s, c \to Nc, t' \to Nt'$ and define the ratio $s = s_2/s_0$. In the genus 0 case the above infinite matrices are conveniently replaced by, [4],

$$Q(1) \to L = \zeta + a_0(x) + \frac{a_1(x)}{\zeta}, \qquad Q(2) \to \tilde{L} = \frac{R(x)}{\zeta} + \sum_{l=0}^{\infty} \frac{b_l(x)}{R^l(x)}\zeta^l$$
$$P^\circ(1) \to M = \frac{x}{\zeta} + \mathcal{O}(\zeta^{-2}), \qquad \widetilde{\mathcal{P}}^\circ(2) \to \tilde{M} = \frac{x}{R}\zeta + \mathcal{O}(\zeta^2)$$

Eq.(17,18) is intepreted as a set of infinitely many equations obtained by expanding in ζ and equating the relevant coefficients.

The equations corresponding to (18) are easy to obtain. The crucial ones are simply (see [4])

$$2t'R + ca_1 = 0, \qquad 2t'b_0 + ca_0 = 0, \qquad x + 2t'b_1 + cR = 0 \tag{22}$$

while the remaining ones determine the unknown elements of $\mathcal{P}^\circ(2)$. From (22) we can determine R, b_0, b_1 in terms of a_0 and a_1. The result is as follows. Corresponding to ζ^q, we have

$$c\frac{b_q}{R^q} = 3t_3 2 \sum_{k=0}^{\infty} \sum_{p=0}^{2k+1} (-1)^k s^{2k+1} \frac{(2k+1)!}{(\frac{p-q}{2})!(\frac{p+q}{2})!(2k+1-p)!} a_0^{2k+1-p} a_1^{\frac{p-q}{2}}, \tag{23}$$

for $q > 1$. The sum over p is limited to the p's such that $p - q$ is even. For $q < -1$ the equations determine the unknown components of $P^\circ(1)$. For $q = 1, 0$ we have, respectively, after replacing (22),

$$\frac{c^2 a_1}{2t'} + a_1 + 6ga_0a_1 - x + 2sg_s \sum_{k=0}^{\infty}\sum_{l=0}^{k} (-1)^k s^{2k} \frac{(2k+1)!}{l!(l+1)!(2k-2l)!} a_0^{2(k-l)} a_1^{l+1} = 0 \tag{24}$$
$$\frac{c^2 a_0}{2t'} + a_0 + 3(a_0^2 g + 2a_1)g + 2sg_s \sum_{k=0}^{\infty}\sum_{2l=0}^{2k+1} (-1)^k s^{2k} \frac{(2k+1)!}{l!l!(2k+1-2l)!} a_0^{2k+1-2l} a_1^{l} = 0. \tag{25}$$

The equation corresponding to ζ^{-1} is a copy of (24) multiplied by a_1. Now we can safely take the decoupling limit $c \to 0$. In this limit $R \sim c$ and $b_l \sim c^{l+1}$, except for b_1 which remains finite. The model splits into a Gaussian model we will henceforth forget about and the cubic+logarithmic model we started with. The latter is determined by the following equations, obtained from (24, 25),

$$a_1 + 6ga_0a_1 - x + 2sg_s \sum_{k=0}^{\infty}\sum_{l=0}^{k} (-1)^k s^{2k} \frac{(2k+1)!}{l!(l+1)!(2k-2l)!} a_0^{2(k-l)} a_1^{l+1} = 0 \tag{26}$$

$$a_0 + 3(a_0^2 g + 2a_1)g + 2sg_s \sum_{k=0}^{\infty}\sum_{2l=0}^{2k+1} (-1)^k s^{2k} \frac{(2k+1)!}{l!l!(2k+1-2l)!} a_0^{2k+1-2l} a_1^{l} = 0 \tag{27}$$

These are the equations one has to solve in order to determine a_0, a_1 and determine all the correlators. We notice that by setting $s = 0$ one gets the equations of the quantum Riemann surface of ref.[4], section 5.1. Therefore, for s small, (26,27) can be thought to represent a deformation of such Riemann surface.

It is not possible to find an exact compact solution of (26,27), as in [4]. However it is rather simple to find solutions in the form of power series, which is just as well for the purpose of determining correlators. For instance, for small x, we easily get

$$a_0(x) = -\frac{6g}{(1+2sg_s)^2}x - 36\frac{9g^3 - g_s g s^3 - 2g_s^2 g s^4}{(1+2g_s s)^5}x^2 + \dots \tag{28}$$

$$a_1(x) = \frac{1}{1+2sg_s}x + 6\frac{6g^2 + g_s s^3 + 2g_s s^4}{(1+2g_s s)^5}x^2 + \dots \tag{29}$$

From such expansions we can compute the correlators of the operators $\tau_k = \mathrm{Tr}(X^k)$ and $\sigma_k = \tilde{q}X^k q$. As for the former, everything works exactly as in [4]. They are obtained by extending the potential as $\mathcal{W}(x) = \sum_{k=1}^{\infty} \hat{t}_k x^k$, differentiating the partition functions with respect to $\hat{t}_k$ and then setting $\hat{t}_k = 0$ except for $\hat{t}_2 = t_2, \hat{t}_3 = t_3$ (which we denote collectively by $\hat{t} = t$). The correlators of the σ_k's can instead be computed in the following way. Let us extend the potential $\mathcal{M}(X)$ to $\hat{\mathcal{M}}(X) = \sum_{k=0}^{\infty} \hat{s}_k X^k$. After differentiating with respect to $\hat{s}_k$ we will set the latter to zero except for $\hat{s}_0 = s_0, \hat{s}_2 = s_2$ (which we denote collectively by $\hat{s} = s$). We obtain

$$\left.\frac{\partial Z}{\partial \hat{s}_k}\right|_{\hat{s}=s} = \int dX dq d\tilde{q} \left.\frac{\partial e^{-S}}{\partial \hat{s}_k}\right|_{\hat{s}=s} = -\int dX\, \mathrm{Tr}\left(\frac{X^k}{\mathcal{M}(X)}\right) e^{\mathrm{Tr}(\mathcal{W}(X) - \ln \mathcal{M}(x))} \tag{30}$$

Expanding

$$\frac{1}{\mathcal{M}(X)} = \frac{1}{s_0}\left(1 - \frac{s}{s_0}X^2 + (\frac{s}{s_0}X^2)^2 - \dots\right)$$

in the integrand, we get

$$< \sigma_n >= -\frac{1}{s_0}\sum_{k=0}^{\infty}(-1)^k \left(\frac{s}{s_0}\right)^k \left.\frac{\partial Z}{\partial t_{2k+n}}\right|_{\hat{t}=t} \tag{31}$$

That is we can express the σ_k correlators as series of the τ_k ones.

5 Conclusions

In this letter we have analyzed topological open string theory on a BPS sector of type IIB string theory on (partly resolved) non–compact Calabi–Yau manifolds and shown that D-branes placed on the blown–down cycles (singular points) produce flavour sectors. From the point of view of gauge theory, the results we have obtained concern the F-terms and therefore our analysis is completely blind to the dynamics of the gauge theory. To summarize, our results amount in practice to the Dijkgraaf-Vafa [20] construction with flavour.

Notice that our picture resembles very much the picture one obtains by considering in type IIA 3 parallel NS 5-branes with D4 suspended between them [5]. Once a lateral 5-brane is moved to infinity, the related gauge sector gets frozen and one is left with matter in the fundamental of the remnant gauge symmetry with flavour symmetry [21]. It would be quite interesting to figure out a detailed string duality connecting the IIA and IIB pictures. This problem was attacked long ago in [16] where it was described as a T-duality. More in general, it would be of some interest to extend our construction to other partially resolved singular toric CY geometries and to be able to read the blow–down operations on the cycles as actions on the related gauge theory quiver diagrams.

The study of the geometric transitions, along the lines of [20, 22], starting from partially resolved CYs might also be of interest in the context of gauge-string correspondence and could shed light on the gauge theory dynamics related to our construction. This goes however beyond the scope of this letter.

Finally let us stress that our construction in section 3 concerns non–linear deformations of the complex structure of the space outside the singularities, while for the singular part we limited ourselves to considering linear deformations (corresponding to the $\tilde{Q}Q$ terms in the effective superpotential). It would be very interesting to know whether such deformations are implied by some geometrical constraint in varying the complex structures of singular spaces. This issue might find a natural explanation in the much more elaborated framework considered in [17, 3].

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Appendix. Partial A_2 resolution

This Appendix is devoted to a nuts–and–bolts description of the partial resolution of a surface with A_2 singularity.

Let us consider the A_2 singular surface given by the equation

$$uv = y^3 \quad \text{where} \quad (u, v, y) \in \mathbf{C}^3. \tag{32}$$

This surface has a doubly singular point at the origin in $\mathbf{C}^3$.

We construct the resolution of this singularity as two copies of $\mathcal{O}(-2)_{\mathbf{P}^1}$ glued along two points by exchanging the base spaces with fibers. In formulas, this implies the overlapping conditions

$$z'_i = z_i^{-1} \quad \text{and} \quad p'_i = z_i^2 p_i \tag{33}$$

[5] These 3 parallel NS 5-branes correspond to the A_2 case. More general cases should correspond to more complicated geometries.

on the intersections of two copies of north and south charts $U_N^i \cap U_S^i$ with $i = 1, 2$; the completely resolved space is then obtained by gluing the south patch U_S^1 and the north patch U_N^2 as

$$z_2' = p_1 \quad \text{and} \quad p_2' = z_1. \tag{34}$$

The blow-down map is given patch by patch by

$$\begin{aligned}
(u, v, y) &= ({p_1'}^2 {z_1'}^3, p_1', p_1' z_1') \quad \text{on} \quad U_N^1 \\
(u, v, y) &= (p_1^2 z_1, p_1 z_1^2, p_1 z_1) \quad \text{on} \quad U_S^1 \\
(u, v, y) &= (z_2'^2 p_2', p_2'^2 z_2', p_2' z_2') \quad \text{on} \quad U_N^2 \\
(u, v, y) &= (p_2, p_2^2 z_2^2, p_2 z_2) \quad \text{on} \quad U_S^2
\end{aligned} \tag{35}$$

The counter-image of the origin in $\mathbf{C}^3$ consists of the union of the two $\mathbf{P}^1$s.

The partial resolution of A_2 is then defined by forgetting one of the two copies of $\mathcal{O}(-2)_{\mathbf{P}^1}$, say the one denoted by $i = 2$, and replacing it with an additional fiber at the singular point. The extra fiber can be recognized upon blow–down to be the fiber sitting at $z_2 = 0$, which is invisible to the other $\mathbf{P}^1$.

Let us denote by A_2' the partially resolved A_2 surface $\mathcal{O}(-2)_{\mathbf{P}^1} \vee \mathbf{C}$. A_2' has a finite volume 2-cycle $\mathbf{P}^1$ and a zero volume 2-cycle pt, the shrunk one, placed at the singular point.

An explicit description of A_2' could go as follows. Let us start from the completely resolved geometry (35) above and define the new coordinates

$$\tilde{u} = p_1, \qquad \tilde{v} = p_1 z_1^2, \qquad \tilde{y} = p_1 z_1 \tag{36}$$

in the chart U_S^1. In the other charts the new coordinates are given by (33,34).

One gets

$$\tilde{u}\tilde{v} = \tilde{y}^2 \tag{37}$$

which is the equation of an A_1 singularity. The counterimage of the singular point $\tilde{u} = \tilde{v} = \tilde{y} = 0$ is the first sphere, parameterized by z_1 and $p_1 = 0$. Therefore this cycle is squeezed to a point in the surface (37). The equations $\tilde{v} = \tilde{y} = 0$ with $\tilde{u}$ generic represents the surviving cycle together with the fiber at $z_1' = 0$. Eq.(37) represents therefore A_2'. For completeness we can proceed to blowing down the second cycle via

$$\hat{u} = \tilde{u}^2 \tilde{v}, \qquad \hat{v} = \tilde{v}, \qquad \hat{y} = \tilde{u}\tilde{v} \tag{38}$$

so that

$$\hat{u}\hat{v} = \hat{y}^2 \tag{39}$$

The inverse image of the singular point $\hat{u} = \hat{v} = \hat{y} = 0$ is precisely the second cycle. We remark that (39), expressed in terms of the coordinates u, v, y, takes the form $y(y^3 - uv) = 0$, which represents the reduced join of the surface (32) with the two original cycles plus extra fibers.

The same construction can be obtained in a neater, although somewhat pedantic way, via toric geometry.

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